1 Context-free grammars

Exercice 1. Give a context-free grammar for each of the following languages (don't justify too much) :

1. $\mathcal{L}_0 = \{a^i b^j c^k \mid i \neq j \vee j \neq k\}$
2. $\mathcal{L}_1 = \{w \in \{a,b\}^* \mid |w|_a \geq |w|_b\}$
3. $\mathcal{L}_2 = \{w \in \{(,),0,1,*,+\} \mid w \text{ is a valid arithmetical expression}\}$

Exercice 2. There are several ways to parse valid arithmetical expression. Find a non-ambiguous grammar where the parsing tree give the priority to $*$ over $+$ (i.e. $2+3*4$ is seen as the expression that equals 14 and not 20).

Exercice 3. Show that the two grammars defined by S and $Balanced$ recognize the same language but one is ambiguous and not the other :

$$\begin{array}{rcl} Balanced & \to & (OneRight\ Balanced \mid \epsilon \\ OneRight & \to &) \mid (OneRight\ OneRight \end{array} \qquad\qquad S \to SS \mid (S) \mid \epsilon$$

2 Context-sensitive grammars

Exercice 4. Let us consider the following grammar :

$$\begin{array}{rcl} S & \to & aSBC + aBC \\ CB & \to & BC \\ aB & \to & ab \end{array} \qquad \begin{array}{rcl} bB & \to & bb \\ bC & \to & bc \\ cC & \to & cc \end{array}$$

What langage does it recognize ? Justify.

Exercice 5. Let us consider the following grammar :

$$\begin{array}{rcl} S & \to & CD \\ C & \to & aCA + bCB \\ AD & \to & aD \\ BD & \to & bD \\ Aa & \to & aA \end{array} \qquad \begin{array}{rcl} Ab & \to & bA \\ Ba & \to & aB \\ Bb & \to & bB \\ C & \to & \epsilon \\ D & \to & \epsilon \end{array}$$

What langage does it recognize ? Justify.

Exercice 6. Let us consider the following grammar :

$$S \to aS + aSbS + \epsilon$$

What langage does it recognize ? Justify.

3 Myhill – Nerode

Let Σ be an alphabet and $\mathcal{L}_\Sigma$ a langage over this alphabet (not necessarily regular).

Definition 1. *Let $x, y \in \Sigma^{*2}$ be two words, we define the following relation :*

$$x \sim_{\mathcal{L}_\Sigma} y \quad \stackrel{\text{def}}{=} \quad \forall z \in \Sigma^* : xz \in \mathcal{L}_\Sigma \Leftrightarrow yz \in \mathcal{L}_\Sigma$$

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Exercice 7. Show that $\sim_{\mathcal{L}_\Sigma}$ defines an equivalence relation (i.e. it is reflexive, symmetric and transitive).

Exercice 8. Let $(x, y) \in \Sigma^{*2}$ show that $x \sim_{\mathcal{L}_\Sigma} y$ implies $xc \sim_{\mathcal{L}_\Sigma} yc$ for all $c \in \Sigma$.

Exercice 9. Show that if there is a finite number of equivalence classes for the relation $\sim_{\mathcal{L}_\Sigma}$ then $\mathcal{L}_\Sigma$ is regular.

Exercice 10. Show that when $\mathcal{L}_\Sigma$ is regular then $\sim_{\mathcal{L}_\Sigma}$ has a finite number of equivalence classes.

4 Myhill – Nerode : applications

Definition 2. *The left quotients of the free monoid induced by $\mathcal{L}_\Sigma$ are the equivalence classes of the relation $\sim_{\mathcal{L}_\Sigma}$.*

The left quotients of $\mathcal{L}_\Sigma$ are the $\mathcal{L}_w$ for $w \in \Sigma^*$ with $\mathcal{L}_w = \{x \mid wx \in \mathcal{L}_\Sigma\}$. Notice there is a finite number of them since $x \sim_{\mathcal{L}_\Sigma} y \Leftrightarrow \mathcal{L}_x = \mathcal{L}_y$.

Exercice 11. Find the left quotient and the minimal automaton for each of the following languages :

- $\mathcal{L}_3 = b(ba)^*$
- $\mathcal{L}_4 = \{a^i b^j \mid i + j \text{ is even}\}$
- $\mathcal{L}_5 = \{w \in \{a, b\}^* \mid w \text{ contains exactly once the factor } bb\}$