1 Are the following languages regular ? (Justify)

Exercice 1. $\mathcal{L}_0 = \{w \in \{a,b,c\}^* \mid (|w|_a = 0) \Rightarrow (|w|_b = 0)\}$

Exercice 2. $\mathcal{L}_2 = \{w \in \{a,b\}^* \mid |w|_a < |w|_b\}$

Exercice 3. $\mathcal{L}_1 = \{w \in \{a,b\}^* \mid 7 \text{ divide } |w|_a, 3 \text{ divide } |w|_b, \}$

Exercice 4. $\mathcal{L}_3 = \{w \in \{(,)\}^* \mid w \text{ is well-parenthesized}\}$

2 Non equivalency of pumping lemmas

Here are three version of the pumping lemma for a language $\mathcal{L}$:

1) $\exists n \in \mathbb{N}, \forall u \in \mathcal{L} : |u| \geq n \quad \Rightarrow \quad \exists v,t,w \in \Sigma^* \quad u = vtw \quad |t| > 0 \quad \forall m \in \mathbb{N} \quad vt^m w \in \mathcal{L}$
2) $\exists n \in \mathbb{N}, \forall rus \in \mathcal{L} : |u| \geq n \quad \Rightarrow \quad \exists v,t,w \in \Sigma^* \quad u = vtw \quad |t| > 0 \quad \forall m \in \mathbb{N} \quad rvt^m ws \in \mathcal{L}$
3) $\exists n \in \mathbb{N}, \forall ru_1 \dots u_n s \in \mathcal{L}, |u_i| \geq 1 \quad \Rightarrow \quad \exists 1 \leq i < j \leq n \quad \forall m \in \mathbb{N} \quad ru_1 \dots u_{i-1}(u_i \dots u_j)^m u_{j+1} \dots u_n s \in \mathcal{L}$

Exercice 5. Show that $\mathcal{L} = \{w \in \{a,b\}^* \mid |w|_a = |w|_b\}$ satisfies lemma 1 but not lemma 2.

Exercice 6. Show that $\mathcal{L} = \{(ab)^n(cd)^n \mid n \in \mathbb{N}\} \cup \Sigma^*(aa+bb+cc+dd+ac+bd)\Sigma^*$ satisfy lemma 2 but not lemma 3.

3 Brzozowski's algorithm

Let $\mathcal{L}_\Sigma$ be a regular language over the alphabet Σ and let $\mathcal{A} =< \mathcal{Q}, \Sigma, \delta, i, \mathcal{F} >$ be a DFA recognizing the language $\mathcal{L}_\Sigma$ such that all the states of $\mathcal{A}$ are accessible from i.

Exercice 7. Show that $Rev(\mathcal{L}_\Sigma) = \{a_n \dots a_1 \mid a_1 \dots a_n \in \mathcal{L}_\Sigma\}$ is recognized by the NFA $Rev(\mathcal{A}) =< \mathcal{Q}, \Sigma, \delta', \mathcal{F}, \{i\} >$ where $\delta'(q,c) = v \Leftrightarrow \delta(v,c) = q$.

Definition 1. *Given an NFA $\mathcal{B} =< \mathcal{Q}_\mathcal{B}, \Sigma, \delta_\mathcal{B}, \mathcal{I}_\mathcal{B}, \mathcal{F}_\mathcal{B} >$ we note $Det(\mathcal{B})$ the DFA obtained by powerset construction of $\mathcal{B}$, We recall $Det(\mathcal{A}) =< 2^{\mathcal{Q}_\mathcal{B}}, \Sigma, 2^{\delta_\mathcal{B}}, \mathcal{I}_\mathcal{B}, \{q \in 2^{\mathcal{Q}_\mathcal{B}} \mid q \cap \mathcal{F}_\mathcal{B} \neq \emptyset\} >$) with $2^{\delta_\mathcal{B}}(e,v) = \{\bigcup_{q \in e} \delta_\mathcal{B}(q,v)\}$.*

Exercice 8. Show that ofr $Det(B) =< 2^{\mathcal{Q}}_{\mathcal{B}}, \Sigma, 2^{\delta_\mathcal{B}}, \mathcal{I}_\mathcal{B}, \{q \in 2^{\mathcal{Q}_\mathcal{B}} \mid q \cap \mathcal{F}_\mathcal{B} \neq \emptyset\} >$ we have the equivalency between $q \in 2^{\tilde{\delta}_\mathcal{B}}(\mathcal{I}_\mathcal{B}, w_1 \dots w_n)$ and the existence of $q_1, \dots, q_n, q_{n+1}$ such that $q_1 \in \mathcal{I}_\mathcal{B}$, $q_{n+1} = q$ et $q_{i+1} \in \delta_\mathcal{B}(q_i, w_i)$.

Definition 2. $leftEquiv(x,y) = \forall z : (zx \in \mathcal{L}_\Sigma) \Rightarrow (zy \in \mathcal{L}_\Sigma)$

Exercice 9. Let $Det(Rev(\mathcal{L}_\Sigma)) =< \mathcal{Q}_d, \Sigma, \delta_d, i_d, \mathcal{F}_d >$. Show that for all $x,y \in (\Sigma^*)^2$ we have :

$$LeftEquiv(x,y) \Rightarrow (\tilde{\delta_d}(i_d,x) = \tilde{\delta_d}(i_d,y))$$

Exercice 10. Deduce from the Myhill–Nerode theorem that $Det(Rev(\mathcal{L}_\Sigma))$ is a minimal automaton recognizing $Rev(\mathcal{L}_\Sigma)$.

We recall the Myhill–Nerode theorem. Given ax language $\mathcal{L}$ we consider the relation of equivalence $RightEquiv(x,y) = \forall z : xz \in Lc \Leftrightarrow yz \in \mathcal{L}$. The Myhill–Nerode theorem states that any DFA recognizing $\mathcal{L}_\Sigma$ contains at least as many states as the number of equivalence classes of $RightEquiv$

Exercice 11. Deduce an algorithm to compute the minimal automaton recognizing $\mathcal{L}_\Sigma$.

Exercice 12. What is the complexity of this algorithm ?