1 AC

Definition 1. *A boolean circuit with n bits of inputs is a Directed Acyclic Graph (DAG) where the leaves are the n nodes representing the input and the output is the only node whose degree is nul and where each internal node is either an $\wedge$ or a $\vee$ (both can have unbounded fan-in) or a $\neg$ (with fan-in one). Given a word $w \in \{0,1\}^n$ and a circuit C whose input size is n, we can evaluate the circuit on w by replacing the i-th leaf with $\top$ when $w_i = 1$ and $\bot$ when $w_i = 0$.*

Definition 2. *We say that a* family of (boolean) circuits $(C_i)_{i\in\mathbb{N}}$ *accepts a language L when for each $n \in \mathbb{N}$, C_n has n bits of input and accepts $\{w \in L \mid |w| = n\}$.*

Definition 3. *We define the class AC^i as the class of decision problem for which there exists a family of boolean circuits $(C_i)_{i\in\mathbb{N}}$ whose size is polynomial in the input and whose depth is bounded $O(log(n)^i)$, i.e. there exists k such that for all $n \in \mathbb{N}$, the depth of C_n is bounded by $log(n^i) \times k$. The size of C_n is bounded by $P(n)$ for some polynomial P.*

Question 1. Justify why we are only interested in the circuits of polynomial size (and not exponential of unbounded) ?

Question 2. The parity problem corresponds to the language $\{w \in \{0,1\}^* \mid |w|_1 \text{ is even}\}$. Show that the *parity problem* is in AC^1.

Question 3. Show that *addition* is in AC^0. In order to do that exhibits a family of circuits of bounded depth accepting the words $w = abc$ with $|a| = |b| = |c|$ and $a + b = c$ (a, b et c are binary number in e.g. little-endian).

Question 4. Show that *multiplication* is in AC^1. Multiplication is accepting $w = abc$ such that $a \times b = c$ and $2|a| = 2|b| = |c|$ where a,b and c are binary numbers.

Question 5. Show that all regular languages over $\{0,1\}$ are in AC^1.

2 NC

Definition 4. *We note NC^i the class of decision problems accepted by a family of boolean circuits $(C_n)_{n\in\mathbb{N}}$ where the arity of gates $\vee$ and $\wedge$ is bounded to two and where the size of circuits is polynomial and the depth is $O(ln(n)^i)$.*

Question 6. Show that for all i we have $NC^i \subseteq AC^i \subseteq NC^{i+1}$.

Question 7. Show that $NC^0 \neq AC^0$.

2.1 TC

Definition 5. *The class TC^0 corresponds to the class of decision problems accepted by circuits where the gates are $\neg$ and threshold gates $T_k(x_1, \dots, x_n)$ (for all k) with $T_k(x_1, \dots, x_n) = |\{i \mid x_i = \top\}| \geq k$*

Question 8. Show that the gates $\wedge$, $\vee$ and $MAJORITY$ can be encoded with a constant depth using where $MAJORITY(x_1, \dots, x_n) = \top$ when half of its inputs are $\top$.

Question 9. Show that TC^0 is equal to the class of problems accepted by circuits where the gates are only $\neg$, $\wedge$, $\vee$, $\neg$ and $MAJORITY$ (all with unbounded fan-ins).

Question 10. Show that $TC^0 \subseteq NC^1$

Question 11. Show that *parity* $\in TC^0$.

Question 12. Show that *multiplication* $\in TC^0$.