# 1 Reducing from 3-SAT to CNF-SAT

A formula CNF (for Conjunctive Normal Form) is a set of variables  $x_1, \ldots, x_n$  and a set of clauses  $C_1, \ldots, C_k$ where each clause is a disjunction of literals (a literal is either a variable x of its negation  $\neg x$ ). For all  $1 \le j \le k$ , we have  $C_j = \bigvee_i v_i^j$  where  $v_i^j = x_l$  or  $v_i^j = \neg x_l$  for a given l.

A CNF formula is satisfiable if it exists an assignation of the variables  $x_1, \ldots, x_n$  to true  $(\top)$  or false  $(\bot)$  such that each clause is satisfied (i.e. for each clause, at least one of these literals is true). From a logical point of view, a CNF formula is a formula of the form :  $\exists x_1, \ldots, x_n \in \{\top, \bot\}^n : \bigwedge_i (\bigvee_i v_i^j)$ .

**Definition 1.** The CNF-SAT problem is to decide whether a CNF formula is satisfiable.

**Definition 2.** The k-SAT problem is to decide the satisfiability of a CNF formula where each clause has to have at most k literals.

**Exercice 1.** Show that the *CNF-SAT* and *k-SAT* problems (for all  $k \in \mathbb{N}$ )) are  $\mathcal{NP}$ .

**Exercice 2.** Show that the formula  $(v_1 \lor v_2 \lor v_3 \lor v_4)$  is satisfiable if and only if  $(v_1 \lor v_2 \lor l) \land (\neg l \lor v_3 \lor v_4)$  is satisfiable (where *l* is a fresh variable, *i.e. l* does not appear in any of the  $v_1, \ldots, v_4$ ).

**Exercice 3.** Show that 3-SAT is  $\mathcal{NP}$ -complete. Start from CNF-SAT (which is  $\mathcal{NP}$ -complete) and show that for each CNF formula  $\varphi$  we can find  $\varphi'$  satisfiable iff  $\varphi$  also is with  $|\varphi'|$  polynomial in  $|\varphi|$  and where each clause of  $\varphi'$  contains at most 3 literals.

**Definition 3.** A DNF formula (for Disjunctive Normal Form) is a set of variables  $x_1, \ldots, x_n$  and a disjunction of clauses where each clause is a conjunction of literals. A DNF formula is thus equivalent to  $\exists x_1, \ldots, x_n \in \{\bot, \top\}^n : \bigvee_j (\wedge_i v_i^j)$ 

**Exercice 4.** Is the DNF-SAT (satisfiability of DNF formula) in  $\mathcal{NP}$ ? in co- $\mathcal{NP}$ ? in  $\mathcal{P}$ ?

## 2 Reducing from 3-SAT to 3-coloring

**Definition 4.** Given a graph G = (V, E), G is k-coloriable if we can color its nodes with k colors such that no two neighboring nodes have the same color. Formally  $\exists (c : V \to \{1, ..., k\}) \ \forall (i, j) \in E : c(i) \neq c(j)$ .

**Exercice 5.** Show that for each k fixed, the k-coloriability is  $\mathcal{NP}$ .

**Exercice 6.** Show that if the k + 1-coloriability problem is in  $\mathcal{P}$  then so is the k-coloriability problem.

Let us show that 3-SAT can be reduced to 3-coloriage. First let us introduce a few gadgets.

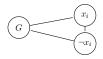
### 2.1 Color gadget

We will make sure that the colors in our graph represent either true  $(\top)$ , false  $(\bot)$  or ground G. To encode these colors, our first gadget is to include in the graph the graph drawn below. After that we will often plug nodes to G or  $\bot$  to forbid colors in nodes. This gadget is only present once in the graph.



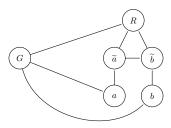
### 2.2 Literal gadget

For each variable  $x_i$  we create two nodes :  $x_i$  and  $\neg x_i$  that we will connected to each other and to G as represented below :



## 2.3 OR gadget

Suppose that a and b represent boolean values (we can enforce that by plugging them into G), the OR gadget is the following (the G node is the one of the color gadget) :



**Exercice 7.** Show that in any valid coloring, the color of R is the color of a or b. And, if we limit the problem to this subgraph then it can be colored with either the color of a or b.

**Exercice 8.** Find a gadget that performs the OR between three boolean values a, b, c (represented as nodes colored either  $\top$  or  $\perp$ ).

### 2.4 Packing everything

**Exercice 9.** For each clause, design a gadget to verify it.

**Exercice 10.** Provide a polynomial reduction from 3-SAT to 3-coloring.

## 3 Miscellaneous

**Exercice 11.** In which classes  $(\mathcal{P}? \mathcal{NP}? \text{ co-}\mathcal{NP}?)$  is the 2-coloring problem?

**Exercice 12.** Show that SAT can be reduced to 3-SAT (i.e the satisfiability of formula composed of  $\lor$ ,  $\land$ ,  $\neg$  and variables).

**Exercice 13.** In which classes is the 2-SAT problem?

## 4 Cliques

**Definition 5.** The clique problem is to decide whether a given graph contains a k-clique (k nodes all direct neighbors of each others).

**Exercice 14.** Show that the clique problem is  $\mathcal{NP}$ -complete.

Suggestion : find a reduction with 3-SAT. Given  $\bigvee_j(\bigwedge_i v_i^j)$  we create a node for each  $v_i^j$  (when a literal appears in several clauses we include it several times).