

1 Context-free grammars

Exercise 1. Give a context-free grammar for each of the following languages (don't justify too much) :

1. $\mathcal{L}_0 = \{a^i b^j c^k \mid i \neq j \vee j \neq k\}$
2. $\mathcal{L}_1 = \{w \in \{a, b\}^* \mid |w|_a \geq |w|_b\}$
3. $\mathcal{L}_2 = \{w \in \{(,), 0, 1, *, +\} \mid w \text{ is a valid arithmetical expression}\}$

Exercise 2. There are several ways to parse valid arithmetical expression. Find a non-ambiguous grammar where the parsing tree give the priority to $*$ over $+$ (i.e. $2 + 3 * 4$ is seen as the expression that equals 14 and not 20).

Exercise 3. Show that the two grammars defined by S and $Balanced$ recognize the same language but one is ambiguous and not the other :

$$\begin{array}{l} \text{Balanced} \rightarrow (\text{OneRight Balanced} \mid \epsilon \\ \text{OneRight} \rightarrow) \mid (\text{OneRight OneRight} \end{array} \qquad S \rightarrow SS \mid (S) \mid \epsilon$$

2 Context-sensitive grammars

Exercise 4. Let us consider the following grammar :

$$\begin{array}{ll} S \rightarrow aSBC + aBC & bB \rightarrow bb \\ CB \rightarrow BC & bC \rightarrow bc \\ aB \rightarrow ab & cC \rightarrow cc \end{array}$$

What language does it recognize? Justify.

Exercise 5. Let us consider the following grammar :

$$\begin{array}{ll} S \rightarrow CD & Ab \rightarrow bA \\ C \rightarrow aCA + bCB & Ba \rightarrow aB \\ AD \rightarrow aD & Bb \rightarrow bB \\ BD \rightarrow bD & C \rightarrow \epsilon \\ Aa \rightarrow aA & D \rightarrow \epsilon \end{array}$$

What language does it recognize? Justify.

Exercise 6. Let us consider the following grammar :

$$S \rightarrow aS + aSbS + \epsilon$$

What language does it recognize? Justify.

3 Myhill – Nerode

Let Σ be an alphabet and \mathcal{L}_Σ a language over this alphabet (not necessarily regular).

Definition 1. Let $x, y \in \Sigma^{*2}$ be two words, we define the following relation :

$$x \sim_{\mathcal{L}_\Sigma} y \stackrel{\text{def}}{=} \forall z \in \Sigma^* : xz \in \mathcal{L}_\Sigma \Leftrightarrow yz \in \mathcal{L}_\Sigma$$

Exercice 7. Show that $\sim_{\mathcal{L}_\Sigma}$ defines an equivalence relation (i.e. it is reflexive, symmetric and transitive).

Exercice 8. Let $(x, y) \in \Sigma^{*2}$ show that $x \sim_{\mathcal{L}_\Sigma} y$ implies $xc \sim_{\mathcal{L}_\Sigma} yc$ for all $c \in \Sigma$.

Exercice 9. Show that if there is a finite number of equivalence classes for the relation $\sim_{\mathcal{L}_\Sigma}$ then \mathcal{L}_Σ is regular.

Exercice 10. Show that when \mathcal{L}_Σ is regular then $\sim_{\mathcal{L}_\Sigma}$ has a finite number of equivalence classes.

4 Myhill – Nerode : applications

Definition 2. The left quotients of the free monoid induced by \mathcal{L}_Σ are the equivalence classes of the relation $\sim_{\mathcal{L}_\Sigma}$.

The left quotients of \mathcal{L}_Σ are the \mathcal{L}_w for $w \in \Sigma^*$ with $\mathcal{L}_w = \{x \mid wx \in \mathcal{L}_\Sigma\}$. Notice there is a finite number of them since $x \sim_{\mathcal{L}_\Sigma} y \Leftrightarrow \mathcal{L}_x = \mathcal{L}_y$.

Exercice 11. Find the left quotient and the minimal automaton for each of the following languages :

- $\mathcal{L}_3 = b(ba)^*$
- $\mathcal{L}_4 = \{a^i b^j \mid i + j \text{ is even}\}$
- $\mathcal{L}_5 = \{w \in \{a, b\}^* \mid w \text{ contains exactly once the factor } bb\}$