1 Arden's rule

Exercice 1. Let $\mathcal{A}$, $\mathcal{B}$ be regular langages, prove that, if $\epsilon \notin \mathcal{A}$ then $\mathcal{X} = \mathcal{A}\mathcal{X} \cup \mathcal{B}$ has a unique language $\mathcal{X}$ solution. Express this solution in terms of a (resp. b) a regular expression recognizing $\mathcal{A}$ (resp. $\mathcal{B}$).

Advice :

1. *find a first solution ;*
2. *show that any solution is larger than your solution ;*
3. *show that any solution is contained within your solution.*

2 Pumping lemma

Exercice 2. Justify whether or not the following languages are regular ?

1. $\{a^n b^n \mid n \in \mathbb{N}\}$
2. $\{a^m b^n \mid n \equiv m \pmod d\}$ for a given $d \in \mathbb{N}$.
3. $\{a^p \mid p$ premier $\}$
4. $\{a^{P(n)} \mid n \in \mathbb{N}\}$ for a given $P \in \mathbb{N}[X]$.

3 Puzzles

Exercice 3. Let x and y be two words such that $xy = yx$. What can you say about the shape of x and y ?

Exercice 4. Let $(p, q) \in \mathbb{N}^*$, and $\mathcal{L}$ a regular language. When is $\frac{p}{q}\mathcal{L} = \left\{u \mid \exists v : uv \in \mathcal{L} \text{ et } |u| = \frac{p}{q}|uv|\right\}$ also a regular language ?

4 Generalized Arden's rule

Definition 1. *A monoid is an algebraic structure* $(E, \bigodot, e)$ *such that :*
(binary operation) For $x, y \in E^2$, $x \bigodot y \in E$
(associative) For $x, y, z \in E^3$, $(x \bigodot y) \bigodot z = x \bigodot (y \bigodot z)$
(identity element) For $x \in E$, $x \bigodot e = e \bigodot x = x$

Definition 2. *A semiring is an algebraic structure* $(E, +, \times, 0, 1)$ *such that :*

- $(E, +, 0)$ *is a commutative monoid* $(a + b = b + a)$ *;*
- $(E, \times, 1)$ *is a monoid ;*
- $\times$ *is distributive over* $+$ $(a \times (b + c) = a \times b + a \times c$ *and* $(b + c) \times a = b \times a + c \times a)$ *;*
- 0 *is absorbing for* $\times$ *(i.e.* $0 \times a = a \times 0 = 0$*)*

Exercice 5. Notice that $(\mathcal{L}_\Sigma, \cup, /, \emptyset, \epsilon)$ is a semiring where :

- $\mathcal{L}_\Sigma$ is the set of regular languages over Σ ;
- $/$ is the concatenation $X/Y = \{xy \mid x \in X, y \in Y\}$;
- $\cup$ is the union ;
- $\emptyset$ is the empty language ;
- ϵ is the language containing only the empty word.

Let $(M_n(\mathcal{L}_\Sigma), +, \times, 0, 1)$ be the set of square matrices of size n over the semiring $(\mathcal{L}_\Sigma, /, \cup, \emptyset, \epsilon)$ with

- $(M + N)_{i,j} = M_{i,j} \cup N_{i,j}$;
- $(M \times N)_{i,j} = \bigcup_{1 \le k \le n} M_{i,k}/N_{k,j}$;
- $0_{i,j} = \emptyset$

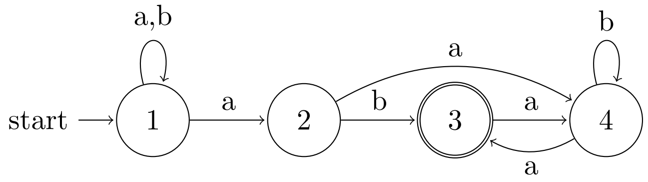
— $1_{i,j} = \begin{cases} \epsilon & i = j \\ \emptyset & \text{sinon} \end{cases}$
— $M^* = \sum_{i \in \mathbb{N}} M^k$
Notice that $(M_n(\mathcal{L}_\Sigma), +, \times, 0, 1)$ is also a semiring.
Remark : any semiring can be transformed into a semiring of matrices

Exercice 6. Let $\mathcal{A} \in M_n(\mathcal{L}_\Sigma)$ with $\forall i, j\ \epsilon \notin A_{i,j}$ and $\mathcal{B} \in M_{n,1}(\mathcal{L}_\Sigma)$ a vector of size n, show that there is a unique solution to $\mathcal{X} \in M_{n,1}(\mathcal{L}_\Sigma)$ with $\mathcal{X} = \mathcal{A}\mathcal{X}^\mathcal{T} \cup \mathcal{B}$ (i.e. $\forall i : \mathcal{X}_i = \left(\bigcup_j\ \mathcal{A}_{i,j}/\mathcal{X}_j\right) \cup \mathcal{B}_i$).

Exercice 7. *(Gaussian elimination)* Given$\mathcal{A} \in M_n(\mathcal{L}_\Sigma)$ and $\mathcal{B} \in M_{n,1}(\mathcal{L}_\Sigma)$ show that when $n > 1$ the languages $(\mathcal{A}^*\mathcal{B})_i$ for $i \in \{1, \ldots, n-1\}$ can be expressed as $\mathcal{A}'^*\mathcal{B}'_i$ for a given $\mathcal{A}' \in M_n(\mathcal{L}_\Sigma), \mathcal{B}' \in M_{n,1}(\mathcal{L}_\Sigma)$.

Exercice 8. Deduce an algorithm returning the regular expression corresponding to a given automata.

Exercice 9. What is the regular expression corresponding to :



5 Monoid of a language

Definition 3. *Given an alphabet Σ and a language $\mathcal{L}$ (not necessarily regular) we state $\mu_\mathcal{L}$ le syntactic morphism from $\mathcal{L}$ defined as : $\mu_\mathcal{L}(u) = \{(x, y) \in \Sigma^{*2} \mid xuy \in \mathcal{L}\}$.*

Exercice 10. Show that $\mu_\mathcal{L}$ is compatible with the concatenation, i.e. $\forall u, v, w : \mu_\mathcal{L}(v) = \mu_\mathcal{L}(w) \Rightarrow (\mu_\mathcal{L}(vu) = \mu_\mathcal{L}(wu)) \wedge (\mu_\mathcal{L}(uv) = \mu_\mathcal{L}(uw))$.

Exercice 11. For each element $e \in \mu_\mathcal{L}(\Sigma^*)$ we chose a an element $r(e)$ tel que $\mu_\mathcal{L}(r(e)) = e$, and we define $u \odot v = \mu_\mathcal{L}(r(u)) \odot \mu_\mathcal{L}(r(v)) = \mu_\mathcal{L}(r(u)r(v))$.
Justify that :
— $\odot$ does not depend on the choices for r.
— the structure $(\mu_\mathcal{L}(\Sigma^*), \odot, \mu_\mathcal{L}(\epsilon))$ induced by $\mu_\mathcal{L}$ is a monoid
and thus $\mu_\mathcal{L}$ do is a morphism from the the free monoid over Σ towards the monoid induced by $\mu_\mathcal{L}$.
Remark : any monoid $(M, \odot, e)$ compatible with a function μ induce a structure of monoid over $\mu(M)$.

Exercice 12. Prove that $\mu_\mathcal{L}(\Sigma^*)$ is finite when $\mathcal{L}$ is regular.

Exercice 13. Prove that when $\mu_\mathcal{L}(\Sigma^*)$ is finite then $\mathcal{L}$ is regular.

Exercice 14. What does the the syntactic monoid of the well-parenthesed words looks like ?